Assignment Note that although these problems are set in $\mathbb{R}^n$, everything in them holds for arbitrary inner product spaces. So your proofs should be based on the properties of inner product spaces, not on specific aspects of $\mathbb{R}^n$. In particular, you should not need to use the standard basis unless it is specifically referred to. Also, be sure to use the basis-free definitions of everything (including $E(X)$ and $C_X$) and not the ones from the first half of the semester.

1. Prove from the definition of vector-valued expected value that if $X$ is a random variable and $w \in \mathbb{R}^n$, then $E(Xw) = E(X)w$. (The $E$ on the left is vector-valued expected value, while the $E$ on the right is the usual real-valued expected value.)

2. Prove that if $Y$ is an $\mathbb{R}^n$-valued random vector that takes all its values in a subspace $V \subset \mathbb{R}^n$, then $E(Y) \in V$.

3. Let $X$ and $Y$ be $\mathbb{R}^n$-valued random vectors with covariance operator $C$. Prove that for all linear functionals $f, g : \mathbb{R}^n \to \mathbb{R}$:

$$
\text{Cov}(f(X), g(Y)) = f(C(g)),
$$

where $g$ is the coefficient vector of $g$.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that

$$
C_{T(Y)} = TC_Y T',
$$

where $C$ indicates the variance operator of the subscripted random vector and $T' : \mathbb{R}^m \to \mathbb{R}^n$ is the adjoint of $T$.

5. Let $p_1, \ldots, p_n$ be strictly positive numbers summing to 1, and let $X = [X_1 \cdots X_n]'$ be an $\mathbb{R}^n$-valued random vector taking the value $e_j$ with probability $p_j$ for $j = 1, \ldots, n$. Define the random vector $Y = [Y_1 \cdots Y_n]'$ by

$$
Y_j = \frac{X_j - p_j}{\sqrt{p_j}}
$$

for $j = 1, \ldots, n$. Prove that:

(a) $E(Y) = 0$.

(b) The variance operator $C_Y$ of $Y$ is projection onto the orthogonal complement of the span of the vector $w = [\sqrt{p_1} \cdots \sqrt{p_n}]'$ (which is also known as the qr projection $Q_w$).

6. Recall that an $\mathbb{R}^n$-valued random vector $Y$ is called nonsingular if $\text{Var}(v \cdot Y) > 0$ for all $v \neq 0$.

Let $Y$ be a nonsingular $\mathbb{R}^n$-valued random vector, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a nonsingular (or, equivalently, invertible) linear transformation. Prove that $T(Y)$ is a nonsingular random vector.

7. Recall that an $\mathbb{R}^n$-valued random vector $Y$ is called weakly spherical if its covariance operator is $\sigma^2 I$ for some nonnegative $\sigma \in \mathbb{R}$ (and $\sigma^2$ is called its variance parameter). Prove that an $\mathbb{R}^n$-valued random vector $Y$ is weakly spherical if and only if, for some $v \in \mathbb{R}^n$, the random vectors $Y - v$ and $T(Y - v)$ have the same variance operator for all orthogonal linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. Hint: First figure out what the vector $v$ ought to be.

8. Prove that if an $\mathbb{R}^n$-valued random vector $Y$ is weakly spherical, then either:

- $Y$ is nonsingular, or
- the value of $Y$ equals some constant vector $w$ with probability 1.

9. Let $Y$ be a weakly spherical $\mathbb{R}^n$-valued random vector with variance parameter $\sigma^2$, and let $V \subset \mathbb{R}^n$ be a subspace. Also, let $X = P_V(Y)$. Show that:

(a) When $X$ is considered as an $\mathbb{R}^n$-valued random vector, its variance operator is $\sigma^2 P_V$.

(b) When $X$ is considered as a $V$-valued random vector, its variance operator is $\sigma^2 I_V$, where $I_V : V \to V$ is the identity transformation.