ASSIGNMENT

1. Let \( W \subset \mathbb{R}^n \) be a subspace. Prove that \( W^\perp \), the orthogonal complement of \( W \), is a subspace of \( \mathbb{R}^n \).

2. Let \( W \subset \mathbb{R}^n \) be a subspace. Prove that \( \text{Proj}_W : \mathbb{R}^n \to \mathbb{R}^n \) is a linear transformation.

3. Suppose that \( \{ v_1, \ldots, v_k \} \) is an orthonormal basis for the subspace \( W \subset \mathbb{R}^n \) which you extend to an orthonormal basis \( \{ v_1, \ldots, v_k, v_{k+1}, \ldots, v_n \} \) for \( \mathbb{R}^n \) (where \( k < n \)). Prove that \( v_{k+1}, \ldots, v_n \) are a basis for \( W^\perp \), the orthogonal complement of \( W \).

4. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation that is reflection across the line \( y = 3x \). Find the matrix \( T_E \) of \( T \) relative to the standard basis.

5. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation that is reflection across the plane in \( \mathbb{R}^3 \) spanned by
\[
\begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}
\] and
\[
\begin{bmatrix}
2 \\
-1 \\
1
\end{bmatrix}
\].

Find the matrix \( T_E \) of \( T \) relative to the standard basis.

6. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation that is rotation about the origin by the angle \( \theta \). Find the matrix \( T_E \) of \( T \) relative to the standard basis.

7. Let \( B, C, D \) be bases for \( \mathbb{R}^n, \mathbb{R}^m, \) and \( \mathbb{R}^p \) respectively. For any linear transformations \( S : \mathbb{R}^m \to \mathbb{R}^n \) and \( T : \mathbb{R}^p \to \mathbb{R}^m \), define the matrix product \( S_{CB}T_{DC} \) to equal \( (ST)_{DB} \), the matrix of the linear transformation that is the composition of \( S \) and \( T \) (relative to the bases \( D \) and \( B \)). Also, suppose that
\[
S_{CB} = \begin{bmatrix}
s_{11} & s_{12} & \cdots & s_{1m} \\
s_{21} & s_{22} & \cdots & s_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
s_{n1} & s_{n2} & \cdots & s_{nm}
\end{bmatrix}
\] and
\[
T_{DC} = \begin{bmatrix}
t_{11} & t_{12} & \cdots & t_{1m} \\
t_{21} & t_{22} & \cdots & t_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
t_{n1} & t_{n2} & \cdots & t_{nm}
\end{bmatrix}
\].

Show that the \( ij \)-th element of the matrix product \( S_{CB}T_{DC} \) equals
\[
\sum_{k=1}^{m} s_{ik}t_{kj}.
\]

(If the notation helps you, you may name the vectors in the \( D \) basis \( d_1, \ldots, d_p \), and similarly for \( C \) and \( B \).) In short, you are being asked to show that, far from being strange or arbitrary, the usual definition of matrix multiplication arises in this natural way (by lifting the definition from the composition of linear transformations).

8. Let \( W \subset \mathbb{R}^n \) be a subspace. Prove that \( W \cap W^\perp = \{ 0 \} \), where \( 0 \) denotes the zero vector. (You should be able to do this without using a basis.)

(As an additional optional exercise not to be turned in, prove that the span of the first \( k \) vectors in a basis \( B \) for \( \mathbb{R}^n \) equals the span of the first \( k \) vectors in the orthonormal basis obtained by applying the Gram-Schmidt process to \( B \), for any \( k \leq n \).)