1. Recall that the $k$-th central moment (or $k$-th moment about the mean) is defined to be

$$E((X - E(X))^k).$$

Let $X$ be a random variable with $X \sim \text{Unif}(a, b)$. Find the $k$-th central moment of $X$ in terms of $a$ and $b$. Simplify your answer separately for when $k$ is even and when $k$ is odd.

2. Prove that $\Gamma(1/2) = \sqrt{\pi}$. (Hint: You might find a change of variables in the integral useful.)

3. Prove that for all $x \in \mathbb{R}$ with $x > 0$,

$$\Gamma(x + 1) = x\Gamma(x)$$

(Hint: You might want to use integration by parts.)

4. A random variable $X$ is said to have a Pareto distribution with parameter $\alpha > 0$ if its probability density function is

$$f(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & \text{for } x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the expected value of a random variable $X$ that has a Pareto distribution with parameter $\alpha > 0$. Give separate answers for when $0 < \alpha \leq 1$ and $\alpha > 1$.

5. (solo problem) Prove that for any positive integer $n$, $\Gamma(n) = (n-1)!$. 