Monday, February 18: Livio, Ch 6

Reading questions for Chapter 6 of Livio (group theory)

Key ideas in Galois’ proof:

- “symmetry profile” of a group: permutations of roots
- normal subgroups
- solvable groups
Symmetry profile:

Roots of $x^2 - 2$ are $\pm \sqrt{2}$

If we extend the rational numbers by these, we are working in a field whose elements look like $a + b\sqrt{2}$

We can transform that field by the identity function and by the function

$$f(a + b\sqrt{2}) = a - b\sqrt{2}$$

This preserves the real numbers

The Galois group of this polynomial is $C_2$
A subgroup is a group within a group

**Theorem** The order of a subgroup divides the order of a group

The quotient of the order of the big over the order of the small is called the **index** of the small

A normal subgroup is a group whose elements *almost* commute with the whole group

That is, $H$ is normal in $G$ if for each $h \in H$ there exists and $h' \in H$ such that $gh = h'g$
In an abelian group, all subgroups are normal.

In $D_3$, \( \{I, R, R^2\} \) is normal, but \( \{I, F\} \) is not (since $FR \neq RF$ and $FR \neq IF$).
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Maximal (proper) normal: normal, and isn’t contained in any larger normal proper subgroups

Any maximal normal subgroup can be extended in a chain down to the identity subgroup

$S_2 : 2, 1$

$S_3 : 6, 3, 1$

$S_4 : 24, 12, 4, 2, 1$

$S_5 : 120, 60, 1$

A group is solvable if it contains a chain of maximal normal subgroups for which every index is a prime number
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This associates an algebraic object with each object of study (polynomial) and then answers questions about the polynomial by answering questions about the algebraic object.
How do we know that the symmetry groups for symmetry types $C_n$ and $D_n$ are not isomorphic (except $C_2$ and $D_1$, where they are isomorphic)?

We can describe the multiplication tables in terms of generators and relations.
A set of transformations \textit{generates} a group if the entire group can be obtained by composing these transformations with each other (as many times as you like)

In this case, the transformations in a set that generates the group are called \textit{generators} of the group

These generators are not unique
Wednesday, February 20: Planar finite figures

For example, the cyclic group $C_n$ is generated by a single element (although the generating element is not unique)

On the other hand, the dihedral group $D_n$ is generated by two elements (again not unique)
Relations are equations satisfied by generators of a group that completely determine the multiplication table of the group.

A listing of the generators and relations describing a group $G$ is called a presentation of the group, often written as:

$$G = \langle \text{generators} \mid \text{relations} \rangle.$$
Wednesday, February 20: Planar finite figures

For example, the cyclic group $C_n$ is generated by $R$ subject to the relation $R^n = I$, so

$$C_n = \langle R \mid R^n = I \rangle.$$ 

Note that this group is \textit{abelian}.

It is not hard to show that the symmetry group for $D_n$ is \textit{nonabelian}.

On the homework, you will find a presentation for the symmetry group for $D_n$. 
Wednesday, February 20: Planar finite figures

A **frieze pattern** is a figure in $\mathbb{R}^2$ whose translational symmetries are generated by a single translation.

A **wallpaper pattern** is a figure in $\mathbb{R}^2$ whose translational symmetries are generated by two translations (but not one).

In a frieze pattern, what symmetries can occur? How many symmetry types are there?
Friday, February 22: Frieze patterns

When translational symmetries are present, other symmetries multiply like rabbits!

For example, if translational symmetries are generated by $T$ and if a single vertical reflection $F$ is present, then so are $TF$ and $T^2F$, etc.
Friday, February 22: Frieze patterns

A useful concept is that of a *fundamental domain*

Suppose we have a frieze pattern $F$, and that $X$ is some part of $F$.

The $X$ is a **fundamental domain** if the following conditions hold:

- The set of all $T(X)$ (where $T$ varies over all the translational symmetries of $F$) is all of $F$.
- The set of all $T(X)$ (where $T$ varies over all the translational symmetries of $F$) do not overlap at all.

As an example, consider the frieze pattern that consists of all the integers on the real line.

It is possible to define fundamental domains related to not only translations, but also other symmetries, but we won’t do so in this class.
Given a fundamental domain, **fundamental symmetry** is a symmetry satisfying one of the following three conditions:

1. It is a rotation and its rotocenter is in the given fundamental domain.
2. It is a reflection and its line of reflection intersects the given fundamental domain.
3. It is an irreducible glide reflection which generates the irreducible glide reflections across that line of reflection, and whose line of reflection intersects the given fundamental domain.

Also, since any irreducible glide reflection whose line of reflection intersects the given fundamental domain has an inverse with the same property, only one of those two should be counted as a fundamental symmetry (you choose which one).

The term *intersects* means *has at least one point in common with*
Friday, February 22: Frieze patterns

Let’s try to figure out all the symmetry types of frieze patterns
The following table gives a classification of the symmetry types of frieze patterns (parentheses means that it is automatic if the non-parentheses items hold):

<table>
<thead>
<tr>
<th>Symmetry Type</th>
<th>Reflection across horizontal?</th>
<th>Reflection across vertical?</th>
<th>180 degree rotation?</th>
<th>Irreducible glide reflection?</th>
</tr>
</thead>
<tbody>
<tr>
<td>p111</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>p1a1</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>p112</td>
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<td>no</td>
<td>yes</td>
<td>(no)</td>
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<tr>
<td>pm11</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>(no)</td>
</tr>
<tr>
<td>pma2</td>
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<td>yes</td>
<td>yes</td>
<td>(no)</td>
</tr>
<tr>
<td>p1m1</td>
<td>yes</td>
<td>no</td>
<td>(no)</td>
<td>(yes)</td>
</tr>
<tr>
<td>pm22</td>
<td>yes</td>
<td>yes</td>
<td>(yes)</td>
<td>(no)</td>
</tr>
</tbody>
</table>
Friday, February 22: Frieze patterns