Wednesday, February 13: Symmetry groups

Recall that a group is a set together with an operation (generically called multiplication) satisfying: (1) multiplication is associative, (2) there is an identity element, and (3) every element has an inverse.

The set of symmetries of a figure, together with the operation of composition, forms a group.

This group is called the symmetry group of the figure.
As we have discussed, the idea of "when are two mathematical objects to be considered "the same"" is an important one in contemporary mathematics.

When are two symmetry groups to be considered "the same"?

This leads us to a new concept...
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Two groups are isomorphic if their multiplication tables are the same except for possibly a relabeling.

We say that two figures have the same symmetry type if their symmetry groups are isomorphic.
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For example, consider the symmetry groups of $T$ and $Z$

The symmetry group of $T$ has the following multiplication table:

\[
\begin{array}{c|cc}
  & I & F_v \\
\hline
I & I & F_v \\
F_v & F_v & I \\
\end{array}
\]

The symmetry group of $Z$ has the following multiplication table:

\[
\begin{array}{c|cc}
  & I & R_{1/2} \\
\hline
I & I & R_{1/2} \\
R_{1/2} & R_{1/2} & I \\
\end{array}
\]

These two symmetry groups are isomorphic, as can be seen by relabeling $F_v$ as $R_{1/2}$ (no reordering is needed)
An observation: if two groups are isomorphic, then they have the same number of elements.

For any element $g$ of a group, its order is the smallest positive $n$ for which $g^n = i$ (where $i$ is the identity).

The order of an element can be infinite (if the group has infinitely many elements).
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The order of I is 1

The order of a reflection is 2

The order of a rotation could be anything: for example, the order of $R_{1/2}$ is 2, the order of $R_{1/3}$ is 3, the order of $R_{3/4}$ is 4, etc.

The order of a nonzero translation or glide reflection is infinite
An observation: if two groups are isomorphic, then their corresponding elements (under a relabeling making them isomorphic) have the same orders.

This means that the square with noses and the letter X do not have the same symmetry type, since the letter X has no symmetries of order 4, but the square with noses does.
What are the symmetries of the following figure (shown in white; the black is not part of the figure)?
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What are the symmetries of the following figure (ignoring all colors)?
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Classification theorem
for figures in $\mathbb{R}^2$ with no translational symmetries

The symmetry group of such a figure satisfies exactly one of the following:

- It consists of the identity $I$ and nothing else
- It consists of exactly 2 elements, in which case the figure’s symmetry type can be called either $C_2$ or $D_1$ (these are the same symmetry type)
- It consists of exactly $n \geq 3$ rotations (including $I$), in which case the figure’s symmetry type is $C_n$
- It consists of exactly $n \geq 3$ rotations (including $I$) and $n$ reflections, in which case the figure’s symmetry type is $D_n$
- It contains an infinite number of symmetries
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The $C_n$ symmetry types (and their corresponding symmetry groups) are called cyclic, and the $D_n$ symmetry types (and their corresponding symmetry groups) are called dihedral.

Notice that $C_n$ consists of $n$ elements, while $D_n$ consists of $2n$ elements.
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For example, the “square with noses” has symmetry type $C_4$, while a square has symmetry type $D_4$.

A circle has an infinite symmetry group.

Notice that the $D_n$ symmetry group has $2n$ symmetries, while the $C_n$ symmetry group has $n$ symmetries.