Notation We denote the real number line by $\mathbb{R}$.

Notation We use the notation $f : A \to B$ (read “$f$ is from $A$ to $B$”) to denote that $f$ is a function whose input is from $A$ and whose output is in $B$.

Notation If $f : A \to B$ and $g : B \to C$, then we denote the composition of these by $gf$, so for all $x$ in $A$:

$$gf(x) = g(f(x)).$$

Note: in $gf$, the function $f$ is applied first, and then $g$ is applied. That is, we apply functions from right to left.
**Notation** The identity function on $A$, denoted by $I : A \to A$ is defined by

$$I(x) = x \quad \text{for all } x \text{ in } A.$$  

Here is what $I : \mathbb{R} \to \mathbb{R}$ does to some points:

Each point $x$ is indicated with an $x$, and $I(x)$ is indicated with a circle $\circ$.

Since this is the identity function the $x$'s and $\circ$'s are all on top of each other.
Wednesday, January 23: 1-dimensional symmetries

**Definition** For any point $p$ in $\mathbb{R}$, the **reflection** (or **flip**) of $\mathbb{R}$ across $p$ is the function $F : \mathbb{R} \to \mathbb{R}$ that sends a point $x$ to the point that is the same distance from $p$ but on the other side.

If $F$ is reflection across $p$, then the red and blue distances are equal.
Definition A vector or directed line segment in $\mathbb{R}$ is a line segment in $\mathbb{R}$ with one of its endpoints designated as the initial point and the other endpoint designated as the final point.

We draw a directed line segment as an arrow whose tail is at the initial point and whose head is at the final point of the line segment.

Two directed line segments are considered the same if they are of the same length and have the same orientation (meaning both are pointing in a negative direction or both are pointing in a positive direction along the real number line $\mathbb{R}$).
Definition For any directed line segment $\vec{s}$, the translation of $\mathbb{R}$ by $\vec{s}$ is the function $T: \mathbb{R} \rightarrow \mathbb{R}$ that sends each point in $\mathbb{R}$ to the final point of a copy of $\vec{s}$ placed with its initial point at the original point.

If $T$ is translation of $\mathbb{R}$ by $\vec{s}$, then each red arrow in the picture depicts $\vec{s}$. 
Definition A rigid transformation (or isometry) of $\mathbb{R}$ is a function $f : \mathbb{R} \to \mathbb{R}$ that preserves distances:

$$\text{Dist}(p, q) = \text{Dist}(f(p), f(q))$$

for all points $p, q$ in $\mathbb{R}$

Theorem Any rigid transformation of $\mathbb{R}$ is either: the identity, a reflection, or a translation
**Theorem** The composition of two rigid transformations is again a rigid transformation.

Why? If \( f, g : \mathbb{R} \to \mathbb{R} \) are rigid transformations, then since \( f \) is a rigid transformation:

\[
\text{Dist}(p, q) = \text{Dist}(f(p), f(q))
\]

Since \( f(p), f(q) \) are also points in the real line, and since \( g \) is a rigid transformation:

\[
\text{Dist}(f(p), f(q)) = \text{Dist}(g(f(p)), g(f(q))).
\]
Friday, January 25: 1-dimensional symmetries

Since the composition of two rigid transformations is again a rigid transformation, and since all rigid transformations of $\mathbb{R}$ are either the identity, a reflection, or a translation:

Composing reflections and/or translations must yield either the identity, a reflection, or a translation.

You’ll investigate this on the next homework assignment
Definition A figure in \( \mathbb{R} \) is a subset of \( \mathbb{R} \).

Definition Two figures are called congruent if there is a rigid transformation taking one to the other.

The idea of congruence is an important one in higher mathematics: when do we consider two mathematical objects of a particular type to be “the same”? 
Definition A symmetry of a figure in $\mathbb{R}$ is a rigid transformation that sends the figure to itself. That is, the figure is indistinguishable before and after the rigid transformation is applied to $\mathbb{R}$.