Definition A **linear operator** is a linear transformation from a vector space to itself.

**Definition** Let $A, B$ be sets, and denote the identity function on each by $I_A : A \to A$ and $I_B : B \to B$. A function $f : A \to B$ is **invertible** if there exists a function $g : B \to A$ such that $fg = id_B$ and $gf = id_A$. This function $g$ is called an **inverse** of $f$, and it is usually denoted by $f^{-1}$. 
Definition Let $A, B$ be sets, and let $f : A \to B$. The function $f$ is called one-to-one if the following statement holds: if $f(x) = f(y)$ then $x = y$.

Definition Let $A, B$ be sets, and let $f : A \to B$. The function $f$ is called onto if the following statement holds: for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

Note that the notion of onto requires a codomain to be onto. If this codomain hasn’t already been specified, then it is important to specify it when discussing whether $f$ is onto, as in: $f$ is onto $B$. 
Proposition Let $A, B$ be sets, and let $f : A \to B$ be a function. Then $f$ is invertible if and only if $f$ is one-to-one and onto.

Proof. For the forwards direction, assume $f$ is invertible. Since $f$ is invertible, it has an inverse $f^{-1} : B \to A$.

To check that $f$ is one-to-one, let $x, y \in A$, and suppose that $f(x) = f(y)$. Applying $f^{-1}$ to both sides of the previous equation gives $x = y$.

To check that $f$ is onto, let $b \in B$. By the definition of an inverse $f(f^{-1}(b)) = b$, so there exists an element of $A$ (namely $f^{-1}(b)$ ) that is mapped to $b$ by $f$. Since $b \in B$ was arbitrary, this holds for all elements of $b$, so $f$ is onto.
For the backwards direction, assume that $f$ is one-to-one and onto, and let $b \in B$. Since $f$ is onto, then there exists an $a \in A$, such that $f(a) = b$. Since $f$ is one-to-one, then $a$ is unique. Define $g(b) = a$. Since $b \in B$ was arbitrary, this defines a function $g : B \to A$.

To see that $g$ is an inverse of $f$, first let $b \in B$. Then $g(b)$ is the unique element of $A$ satisfying $f(g(b)) = a$. Since $b \in B$ was arbitrary, this holds for all elements of $B$, so $fg = I_B$.

Next, let $a \in A$. Then $g(f(a))$ is the unique element of $A$ with the property that $f(g(f(a))) = f(a)$. Since $a$ has this property, then $g(f(a)) = a$. Since $a \in A$ was arbitrary, this holds for all elements of $A$, so $gf = I_A$.

This means that $g$ is an inverse of $f$, so $f$ is invertible. ■
**Proposition** Let $A, B$ be sets, and let $f : A \rightarrow B$ be a function. If $f$ is invertible, then its inverse is unique.

**Proof.** Suppose $g, h : B \rightarrow A$ are both inverses of $f$, and let $b \in B$. Since $f : A \rightarrow B$ is invertible, then $f$ is onto $B$. By the definition of onto, this means there exists $a \in A$ such that $f(a) = b$. Applying $g$ to both sides gives that $a = g(b)$. Applying $h$ to both sides gives that $a = h(b)$, which means that $g(b) = h(b)$. Since $b \in B$ was arbitrary, this holds for all elements of $B$, which means that $g = h$. ■
**Definition** An $n \times n$ matrix $A \in \mathbb{R}^{n \times n}$ is invertible if there exists a matrix $B \in \mathbb{R}^{n \times n}$ such that $AB = BA = I$. In this case, the matrix $B$ is called the inverse of $A$ and is usually denoted by $A^{-1}$. 
Wednesday, November 14: Linear operators

**Proposition** Let $V$ be a finite-dimensional real vector space, and let $T: V \to V$ be a linear operator. Also, let $\mathcal{B}, \mathcal{C}$ be ordered bases for $V$. Then $T$ is invertible if and only if $T_{\mathcal{C},\mathcal{B}}$ is invertible.

**Proof.** For the forwards direction, assume that $T$ is invertible. By the definition of invertible, there exists a linear operator $T^{-1}: V \to V$ such that

$$T T^{-1} = I \quad \text{and} \quad T^{-1} T = I.$$

The first equation implies that

$$T_{\mathcal{C},\mathcal{B}} (T^{-1})_{\mathcal{B},\mathcal{C}} = I_{\mathcal{C},\mathcal{C}} = I.$$

The second equation implies that

$$(T^{-1})_{\mathcal{B},\mathcal{C}} T_{\mathcal{C},\mathcal{B}} = I_{\mathcal{B},\mathcal{B}} = I,$$

so $T_{\mathcal{C},\mathcal{B}}$ is invertible, with its inverse being $(T^{-1})_{\mathcal{B},\mathcal{C}}$. 
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For the backwards direction, denote $T_{C,B} \in \mathbb{R}^{n \times n}$ by $T$. This means that $T_{C,B} = T$.

Suppose that $T$ is invertible. By the definition of invertible, there exists a matrix $T^{-1} \in \mathbb{R}^{n \times n}$ satisfying

$$TT^{-1} = I \text{ and } T^{-1}T = I.$$  

Applying inverse representations to both sides of the first equation gives that

$$T(T^{-1})^{B,C} = I.$$  

Doing the same with the second equation gives that

$$(T^{-1})^{B,C}T = I,$$

so $T$ is invertible, with inverse $(T^{-1})^{B,C}$. ■
Proposition Let $V$ be a finite-dimensional real vector space, and let $T : V \to V$ be a linear operator. Then the following are equivalent:

1. $T$ is invertible
2. $T_{C,B}$ is invertible for some ordered bases $B, C$ of $V$
3. $T_{B,B}$ is invertible for some ordered basis $B$ of $V$
4. $\text{Ker}(T) = \{0\}$
5. $\text{Range}(T) = V$.

Proof. Exercise for the reader.
How can we determine whether $T^{-1}$ exists, and find it if it does?

Well, first choose an ordered basis $B = (b_1, \ldots, b_n)$ of $V$. Then $S$ is an inverse of $T$ if and only if

$$T(S(b_j)) = b_j \quad \text{for all } j = 1, \ldots, n.$$ 

This gives $n$ matrix equations, one for each basis vector, where we are solving for $x_j = S(b_j)$. Note that the solution $(x_j)_B$ is the $j$-th column of $S_{B,B}$. 

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Representing both sides of the above equation relative to $\mathcal{B}$, the $n$ matrix equations that we’d like to solve are

$$T_{\mathcal{B},\mathcal{B}}\mathbf{x}_{j\mathcal{B}} = \vec{e}_j,$$

for $j = 1, \ldots, n$.

That is, we’d like to solve the augmented matrix system of equations

$$\begin{bmatrix} T_{\mathcal{B},\mathcal{B}} & \vec{e}_j \end{bmatrix}$$

for each $j = 1, \ldots, n$.

Since the Gaussian elimination to the left of the $|$ is exactly the same no matter what $j$ is, we can solve all of these matrix systems of equations at once by solving the super-augmented system of equations

$$\begin{bmatrix} T_{\mathcal{B},\mathcal{B}} & I \end{bmatrix},$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.
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When solving

$$\begin{bmatrix} T_{B,B} & I \end{bmatrix},$$

to determine the invertibility of $T$, apply Gaussian elimination to arrive at the reduced row echelon form of the part to the left of the $|$. Then:

1. These equations will all be solvable precisely when the reduced row echelon form to the left of the $|$ is the identity matrix. So $T$ is invertible precisely when this is the case.

2. If the reduced row echelon form to the left of the $|$ is the identity matrix, then whatever remains to the right of the $|$ is $(T^{-1})_{B,B}$.

By a similar argument, this process can be used to find the inverse of a matrix without worrying about what linear transformation it represents.
Proposition Let $T \in \mathbb{R}^{n \times n}$. Then $T$ is invertible if and only if $\det T \neq 0$.

Proof. Exercise for the reader. (Hint: One approach is to recall how the determinant can be computed using Gaussian elimination.) ■
Proposition Let $S, T \in \mathbb{R}^{n \times n}$. Then

$$\det AB = \det A \det B.$$  

Proof. Beyond the scope of this class.
Proposition Let $T \in \mathbb{R}^{n \times n}$ be invertible. Then
\[
\det T^{-1} = \frac{1}{\det T}.
\]

Proof. By the definition of the inverse of a matrix,
\[
TT^{-1} = I.
\]
Taking the determinant of both sides gives
\[
\det TT^{-1} = 1.
\]
Applying the proposition above that the product of the determinant is the determinant of the product gives that
\[
\det T \det T^{-1} = 1.
\]
Since $T$ is invertible, then $\det T \neq 0$. Dividing both sides of the above equation by this proves the proposition. □
Friday, November 16: Linear operators

Let $V$ be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Also, let $\mathcal{B}, \mathcal{C}, \mathcal{C}'$ be ordered bases of $V$

Since $T = IT$, then

$$T_{\mathcal{C}', \mathcal{B}} = I_{\mathcal{C}', \mathcal{C}} T_{\mathcal{C}, \mathcal{B}}.$$

(Changing the input basis can be handled similarly by putting the identity transformation on the left.)

Note that the columns of $I_{\mathcal{C}', \mathcal{C}}$ are the representations of $I(c_i) = c_i$ relative to $\mathcal{C}'$

So to find $I_{\mathcal{C}', \mathcal{C}}$, we need to figure out how to represent the vectors in $\mathcal{C}'$ relative to $\mathcal{C}$
Also note that for an invertible linear transformation, $TT^{-1} = \mathbf{I}$ implies $T_{C,B}(T^{-1})_{B,C} = \mathbf{I}$, so the second matrix is the inverse of the first.

This means that $I_{C,B}$ and $I_{B,C}$ are inverses of each other.
So for $\mathbf{T} : \mathcal{V} \rightarrow \mathcal{V}$, we have

$$\mathbf{T}_{\mathcal{C}, \mathcal{C}} = \mathbf{I}_{\mathcal{C}, \mathcal{B}} \mathbf{T}_{\mathcal{B}, \mathcal{B}} \mathbf{I}_{\mathcal{B}, \mathcal{C}}$$

$$= \mathbf{I}_{\mathcal{C}, \mathcal{B}} \mathbf{T}_{\mathcal{B}, \mathcal{B}} (\mathbf{I}_{\mathcal{C}, \mathcal{B}})^{-1}$$

Taking the determinant of both sides, and using that the determinant is multiplicative, we get

$$\det(\mathbf{T}_{\mathcal{C}, \mathcal{C}}) = \det(\mathbf{I}_{\mathcal{C}, \mathcal{B}} \mathbf{T}_{\mathcal{B}, \mathcal{B}} (\mathbf{I}_{\mathcal{C}, \mathcal{B}})^{-1})$$

$$= \det(\mathbf{I}_{\mathcal{C}, \mathcal{B}}) \det(\mathbf{T}_{\mathcal{B}, \mathcal{B}}) \det((\mathbf{I}_{\mathcal{C}, \mathcal{B}})^{-1})$$

$$= \det(\mathbf{I}_{\mathcal{C}, \mathcal{B}}) \det(\mathbf{T}_{\mathcal{B}, \mathcal{B}}) \frac{1}{\det(\mathbf{I}_{\mathcal{C}, \mathcal{B}})}$$

$$= \det(\mathbf{T}_{\mathcal{B}, \mathcal{B}}).$$
This proves the following:

**Proposition** Let $V$ be a finite-dimensional real vector space, let $T : V \to V$ be a linear operator, and let $\mathcal{B}, \mathcal{C}$ be ordered bases of $V$. Then

$$T_{\mathcal{B},\mathcal{B}} = T_{\mathcal{C},\mathcal{C}}.$$

That is, the determinant of linear operator is same no matter what basis is chosen (as long as input and output bases are the same).

**Definition** Let $V$ be a finite-dimensional real vector space, and let $T : V \to V$ be a linear operator. The determinant of $T$, denoted by $\det(T)$, is defined to be $\det(T_{\mathcal{B},\mathcal{B}})$, where $\mathcal{B}$ is any ordered basis of $V$. 
Corollary. There many equivalent things for $n$ vectors in $\mathbb{R}^n$: 

1. form a linearly independent set
2. form a matrix with nonzero determinant
3. determine a hyperparallelapiped with nonzero oriented volume
4. coefficient nullspace is the zero set
5. echelon set is the full set
6. span is $n$-dimensional
7. reduced row echelon form is $I$
8. kernel of linear operator that they represent is $\{0\}$
9. range of linear operator that they represent is $\mathbb{R}^n$
10. represent a one-to-one linear operator
11. represent an onto linear operator
12. represent an invertible linear operator