Let 
\[ \mathcal{F} = \{ \text{piecewise continuous } f : \mathbb{R} \to \mathbb{C} \mid f(x + 2\pi) = f(x) \}, \]
where “piecewise continuous” here means that there are only finitely many points of discontinuity within any interval of length $2\pi$, and that limits from the left side and from the right side exist at all the points of discontinuity.

In other words, $\mathcal{F}$ is the set of all piecewise continuous complex-valued periodic functions on $\mathbb{R}$ whose periods divide evenly into $2\pi$. To express this even more compactly, these spaces consist in essence of reasonably well-behaved $2\pi$-periodic functions. It is readily verified that, with the usual addition and scalar multiplication of functions, $\mathcal{F}$ is a complex vector space. This vector space is central to the study of Fourier series (hence the letter $\mathcal{F}$ in the notation). We can make $\mathcal{F}$ into a complex inner product space with the following inner product:

\[ \langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \, dx, \]

It is not hard to verify that this function satisfies the definition of an inner product on $\mathcal{F}$, but you don’t need to do so here.

1. Consider the following three functions $f_1, f_2, f_3 \in \mathcal{F}$, defined as follows for $0 \leq x < 2\pi$ and extended periodically to all of $\mathbb{R}$:

\[
\begin{align*}
    f_1(x) &= x \\
    f_2(x) &= x^4 \\
    f_3(x) &= \cos(2x).
\end{align*}
\]

Is the set \{f_1, f_2, f_3\} linearly independent?

2. Show that

\[ \langle e^{ipx}, e^{iqx} \rangle = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases} \]

In other words, show that

\[ \mathcal{A} = \{ e^{ipx} \mid p \in \mathbb{Z} \} \]

is an orthonormal set.

3. It turns out that the set $\mathcal{A}$ from the previous problem is actually a basis for $\mathcal{F}$. In this problem, you may ignore the complications that arise from convergence issues for infinite sums and assume that what you know about bases of finite-dimensional complex vector spaces carries over to bases of infinite-dimensional complex vector spaces.

With this in mind, let $f$ be the function which has period $2\pi$ and is given by

\[
    f(x) = \begin{cases} 
        1 & \text{if } 0 \leq x < \pi \\
        -1 & \text{if } \pi \leq x < 2\pi
    \end{cases}
\]

on the interval $[0, 2\pi)$. 
Use the orthonormal technique to compute the complex Fourier coefficients of \( f \). That is, find the constants \( c_k \) for which

\[
f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.
\]

(The convergence here is pointwise as it turns out, at least away from its discontinuities, but you may ignore such complications here.)

4. Let \( n \) be a positive integer, and let \( x_1, \ldots, x_n \in \mathbb{R} \). Prove that

\[
\left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} x_i^2.
\]

Another way to state this as that the square of their mean is less than or equal to the mean of their squares.

*Hint:* Apply the Cauchy-Schwarz inequality to the dot product of two particular vectors in \( \mathbb{R}^n \).

5. Let

\[
\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}.
\]

Find \( \begin{bmatrix} 1 \\ 49 \\ 9 \\ 29 \end{bmatrix} \mathcal{B} \). *(Hint: There is a long way to do this problem, and a short way. You are welcome to use the long way, but even if you do, you should make sure that you know how to do it in the short way, since that is the main point of this problem.)*