1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the vector $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Without writing down a matrix representation of $T$, explain geometrically what the eigenvalues of $T$ are and what the associated eigenspaces of those eigenvalues are.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by
   
   $$T_{\mathcal{E},\mathcal{E}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   By the spectral theorem for self-adjoint operators, $T$ has an eigenbasis. Find an ordered eigenbasis $\mathcal{B}$, stating the eigenvalue for each eigenvector in $\mathcal{B}$. Also, find $I_{\mathcal{E},\mathcal{E}}$ and $I_{\mathcal{B},\mathcal{E}}$ (and check for yourself that $T_{\mathcal{B},\mathcal{B}} = I_{\mathcal{B},\mathcal{E}}T_{\mathcal{E},\mathcal{E}}I_{\mathcal{E},\mathcal{B}}$).

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by
   
   $$T_{\mathcal{E},\mathcal{E}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

   (a) Find all the eigenvalues of $T$ and their associated eigenspaces. Does $T$ have an eigenbasis? If so, find $T_{\mathcal{B},\mathcal{B}}$ for an ordered eigenbasis $\mathcal{B}$. If not, explain why not.

   (b) Repeat Part (a) for $T$ viewed as a linear transformation $T : \mathbb{C} \to \mathbb{C}$.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by
   
   $$T_{\mathcal{E},\mathcal{E}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

   (a) Find all the eigenvalues of $T$ and their associated eigenspaces. Does $T$ have an eigenbasis? If so, find $T_{\mathcal{B},\mathcal{B}}$ for an ordered eigenbasis $\mathcal{B}$. If not, explain why not.

   (b) Repeat Part (a) for $T$ viewed as a linear transformation $T : \mathbb{C} \to \mathbb{C}$.

5. Let $\mathcal{P}_2$ be the real vector space of polynomials in $x$ of degree at most 2. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be differentiation with respect to $x$, so
   
   $$T(p(x)) = p'(x).$$

   We have seen that $T$ is a linear transformation. Find all its eigenvectors and their associated eigenspaces. Does $T$ have an eigenbasis?

6. Let $V$ be the real vector space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. You don’t need to prove it here, but differentiation $T : V \to V$ is a linear operator on $V$:
   
   $$T(f(x)) = f'(x)$$

   for all infinitely differentiable $f : \mathbb{R} \to \mathbb{R}$. Using what you know from calculus about differentiation, find all the eigenvalues of $T$ and give their associated eigenspaces.