Wednesday, January 31: 2-dimensional rigid transformations

Summary so far: a rigid transformation of $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that preserves distances (and also angles, in fact)

Every rigid transformation of $\mathbb{R}^2$ is one of the following types:

1. translation
2. rotation
3. reflection
4. glide reflection

The identity transformation can be considered a translation (by a directed line segment of length 0) or a rotation (about any center through angle 0)
For any function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, a fixed point is a point $p$ for which $f(p) = p$, meaning $p$ doesn’t go anywhere when $f$ is applied.

We can classify rigid transformations to some extent by their fixed points:

- Every point in $\mathbb{R}^2$ is a fixed point of the identity transformation.
- Reflections in $\mathbb{R}^2$ have a line of fixed points (the line of reflection).
- Rotations of $\mathbb{R}^2$ have exactly one fixed point (the center of rotation).
- Nonzero translations and glide-reflections have no fixed points.
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- Nonzero translations and glide-reflections have no fixed points.
A rigid transformation of $\mathbb{R}^2$ is orientation-preserving if it does not change the clockwise/counterclockwise sense of angles; otherwise it is called orientation-reversing.

Orientation-reversing rigid transformations send a forwards “R” to a backwards “R”, and vice versa.

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We can classify rigid transformations of $\mathbb{R}^2$ by what they do to the orientation of $\mathbb{R}^2$:

- The identity, translations, and rotations are orientation-preserving.
- Reflections and glide-reflections are orientation-reversing.
We can now begin to use these rigid transformations.

A major concept in modern mathematics is that when working with “things” (elements of sets), it is important to define precisely when two such things are considered “the same,” or in mathematical terms, equivalent.

In geometry, we often use the term congruent rather than equivalent, but how do we define this?
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Recall that a figure in $\mathbb{R}^2$ is a subset of $\mathbb{R}^2$.

We define two figures to be equivalent (or congruent) if there is an orientation-preserving rigid transformation taking one to the other.
A symmetry of a figure in $\mathbb{R}^2$ is a rigid transformation of $\mathbb{R}^2$ that leaves the figure indistinguishable before and after the transformation is applied.

A simple but important observation: the composition of two symmetries of a figure is again a symmetry of the figure.

This means that composition can be considered an operation that is defined on the set of symmetries of a figure.

Slightly more precisely, an operation is a way to combine two elements of a set to get an element of the set (here we “combine” two symmetries to get a symmetry).
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Let's list the symmetries of a capital X:
They are: identity $I$, reflection across the horizontal $F_h$, reflection across the vertical $F_v$, rotation by 180 degrees $R_{1/2}$
What are the symmetries of this square with noses?
Rotations by multiples of 90 degrees: $I = R_0, R_{1/4}, R_{1/2}, R_{3/4}$
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Are these two figures symmetric in “the same way”? 

They have the same number of symmetries, but still it doesn’t look like they are

For this we need a further concept...
Some other observations about composing symmetries:

- Composition is associative: \( f(gh) = (fg)h \) for all symmetries \( f, g, h \)
- The identity transformation \( I \) has the property that \( If = f \) and \( fI = f \) for all symmetries \( f \)
- Every symmetry \( f \) has an \textbf{inverse} symmetry \( f^{-1} \) that “undoes” what it “does”, meaning \( ff^{-1} = I \) and \( f^{-1}f = I \)

Also worth noting is that composing symmetries is not \textbf{commutative}: \( fg \) might be different from \( gf \)
In mathematics, a group is a set with an operation (generically called “multiplication”) that satisfies three properties:

1. The operation is associative
2. There is an identity element $i$ in the set with the property that $if = f$ and $fi = f$ for all $f$ in the set
3. Every element $f$ in the set has an inverse $f^{-1}$ in the set, which has the property that $ff^{-1} = i$ and $f^{-1}f = i$

If the operation is commutative, then the group is called abelian
For a finite group, we can write out a multiplication table, which tells how to compose all pairs of symmetries.

We write them as we read them: the leftmost column entry indicates the element composed on the left (meaning second), and the top row entry indicates the element composed on the right (meaning first).