Wednesday, January 24: 2-dimensional rigid transformations

We denote the Euclidean plane by $\mathbb{R}^2$
A directed line segment in $\mathbb{R}^2$ is a line segment in $\mathbb{R}^2$ with one of its endpoints designated as the initial point and the other endpoint designated as the final point.

We draw a directed line segment as an arrow whose tail is at the initial point and whose head is at the final point of the line segment.
Two directed line segments are considered the same if they point in the same direction along parallel lines and have the same length.

The black arrows represent the same directed line segments below.
For any directed line segment $\vec{s}$, the **translation of $\mathbb{R}^2$ by $\vec{s}$** is the function $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that sends each point in $\mathbb{R}^2$ to the final point of a copy of $\vec{s}$ placed with its initial point at the original point.
Notice that all points are moved by the same amount in the same direction.
For any line \( L \) in \( \mathbb{R}^2 \), the **reflection of \( \mathbb{R}^2 \) across \( L \)** is the function \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) that sends each point \( x \) in \( \mathbb{R}^2 \) to the unique point that:

- lies on the unique line through \( x \) that is perpendicular to \( L \),
- is the same distance (along this perpendicular line) as \( x \) is from \( L \),
- is on the other side of \( L \).
Notice that in each case the distances on the two sides of the reflection line are equal, and that the red line segments are perpendicular to the reflection line.
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A **glide reflection** of $\mathbb{R}^2$ is a function $G : \mathbb{R}^2 \to \mathbb{R}^2$ that is a translation of $\mathbb{R}^2$ followed by a reflection of $\mathbb{R}^2$ across a line *that is parallel to the direction of translation*
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For any point $p$ in $\mathbb{R}^2$ and any angle $a$, the **rotation of $\mathbb{R}^2$ about $p$ by $a$** is the function $R : \mathbb{R}^2 \to \mathbb{R}^2$ that sends each point $x$ in $\mathbb{R}^2$ to the unique point that:

- lies on the unique ray making an angle $a$ with the ray emanating from $p$ and passing through $x$, and
- is the same distance from $p$ as $x$ is.

If clockwise or counterclockwise is not specified for the angle, we take *counterclockwise* to correspond to positive angles, and *clockwise* to correspond to negative angles.
Notice that the distance to the center of rotation is preserved: the red line segments at the two ends of the blue circular arcs have the same length.
A **rigid transformation** of \( \mathbb{R}^2 \) is a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that preserves distances:

\[
\text{Dist}(p, q) = \text{Dist}(f(p), f(q))
\]

for all points \( p, q \) in \( \mathbb{R}^2 \)

By a particular mathematical theorem, any rigid transformation of \( \mathbb{R}^2 \) is either

- a translation,
- a reflection,
- a glide reflection,
- a rotation, or
- the identity (which could be considered either a rotation by 0 degrees or a translation by a directed line segment with length 0)
Another theorem states that the composition of two rigid transformations is again a rigid transformation.

Why? If $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are rigid transformations, then since $f$ is a rigid transformation:

$$\text{Dist}(p, q) = \text{Dist}(f(p), f(q))$$

Since $f(p), f(q)$ are also points in the real line, and since $g$ is a rigid transformation:

$$\text{Dist}(f(p), f(q)) = \text{Dist}(g(f(p)), g(f(q))).$$
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Since the composition of two rigid transformations is again a rigid transformation, and since all rigid transformations of $\mathbb{R}^2$ are either the identity, a translation, a reflection, a glide reflection, or a rotation:

In $\mathbb{R}^2$, composing translations, reflections, glide reflections, and/or rotations must yield either the identity, a translation, a reflection, a glide reflection, or a rotation

You’ll investigate this on the next homework assignment