Homework 8

E1. Let $\ell$ be the line in $\mathbb{R}^2$ described by $y = 2x$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows. Given any $\vec{x} \in \mathbb{R}^2$, there is a unique line through $\vec{x}$ that is perpendicular to the line $\ell$. Define $T(\vec{x})$ to be the unique point on that perpendicular line that is the same distance as $\vec{x}$ from $\ell$ but on the other side of $\ell$. You don’t need to prove it here, but it turns out that $T$ is a linear transformation. Find $T_{E, E}$.

E2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as in the previous problem, and let $B = (\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix})$. Find $T_{B, B}$. (Think about what this matrix is telling you geometrically, although you don’t need to write anything about that here.)

E3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as in the previous two problems. Find a basis for $\text{Ker}(T)$ and a basis for $\text{Range}(T)$, and use this to verify that the Rank-Nullity Theorem does indeed hold for $T$.

E4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$T_{E, E} = \begin{bmatrix}
1/2 & -\sqrt{3}/2 & 0 & 0 \\
\sqrt{3}/2 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & \sqrt{3}/2 \\
0 & 0 & -\sqrt{3}/2 & 1/2
\end{bmatrix}.$$  

Compute $(T^3)_{E, E}$. Describe what $T$ does geometrically. (Note the “block form” of the above representation of $T$, meaning the way that there are nonzero smaller matrices, $2 \times 2$ ones in this case, along the diagonal).

E5. We know what the representation of a rotation by $\pi/2$ about the origin in $\mathbb{R}^2$ looks like relative to the standard basis for $\mathbb{R}^2$ as both input and output basis. Now consider the same rotation in $\mathbb{C}$ (which has, after all, the same underlying set, just different scalars). What is its representation relative to the standard basis for $\mathbb{C}$ as both input and output basis, considering $\mathbb{C}$ as a complex vector space? How about for a rotation of $\mathbb{C}$ by an arbitrary angle $\theta$, using the standard basis for $\mathbb{C}$ and considering $\mathbb{C}$ as a complex vector space?
E6. Let $V$ be the vector space of infinitely differentiable (a.k.a. smooth) functions from $\mathbb{R}$ to $\mathbb{R}$, with the usual definitions of scalar multiplication and addition. Define a linear transformation $T : V \to V$ by

$$(T(f))(t) = f''(t) - 5f'(t) + 6f(t).$$

(In the lingo, one might write $T = \frac{d^2}{dt^2} - 5\frac{d}{dt} + 6$ and refer to $T$ as a differential operator.) Verify in your mind (do not write it up as part of the problem) that $T$ is indeed a linear transformation from $V$ to $V$.

(a) The general theory will tell us that $\text{Dim}(\ker(T)) = 2$. Given this, find a basis for the kernel, starting from the educated guess that it will consist of functions of the form $e^{rt}$ for some $r \in \mathbb{R}$.

(b) Verify that $f(t) = te^t$ is a solution to the equation

$$f''(t) - 5f'(t) + 6f(t) = -3e^t + 2te^t.$$

(c) Use Parts (a) and (b) to find all solutions to the differential equation given in Part(b).

E7. The three Pauli spin matrices are

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(Sometimes denoted by $\sigma_x$, $\sigma_y$, and $\sigma_z$ instead.) You don’t need to prove it, but the set $\mathbb{C}^{2 \times 2}$ of $2 \times 2$ complex matrices forms a complex vector space under its usual matrix operations. What is the dimension of $\text{Span}\{\sigma_1, \sigma_2, \sigma_3\}$ in this complex vector space?