Homework 7

E1. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be reflection across the \( y \) axis, which means that \( T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -x \\ y \end{bmatrix} \).

Prove, directly from the definition of a linear transformation, that \( T \) is a linear transformation.

E2. With \( T \) as before, find \( T(e_1)_E \), \( T(e_2)_E \) and \( T_{E,E} \).

E3. With \( T \) as before, let

\[
B = (b_1, b_2) = (\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}).
\]

Find \( T(b_1)_E, T(b_2)_E \) and \( T_{E,B} \).

E4. With \( T \) and \( B \) as before, find \( T(b_1)_B, T(b_2)_B \) and \( T_{B,B} \). (Notice that the matrix representation of a linear transformation depends very much on what ordered bases you choose!)

E5. Let \( P_k \) be the vector space of polynomials in \( y \) of degree at most \( k \), and let \( T : P_3 \to P_2 \) be differentiation with respect to \( y \):

\[
T(p(y)) = p'(y)
\]

Prove, directly from the definition of a linear transformation, that \( T \) is a linear transformation.

E6. You don’t need to prove it here, but \( B = (1, y, y^2, y^3) \) is a basis for \( P_3 \), and \( C = (1, y, y^2) \) is a basis for \( P_2 \). With \( T \) as in the previous problem, find \( T_{C,B} \).

E7. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation given by:

\[
T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} -2x - 2y + 4z \\ x + 2y - 5z \\ -2x - y + z \end{bmatrix}.
\]

Use Gaussian elimination to find all \( v \in \mathbb{R}^3 \) such that \( T(v) = \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \).