Interlude: complex numbers

We present here an introduction to complex numbers. This introduction is not intended to be a comprehensive one by any means, but rather simply to present enough fundamentals to allow the reader to use complex numbers in linear algebra and thereby to begin to acquire some proficiency with them and be able to explore them further.

Section 2.9: The algebra of complex numbers

The set $\mathbb{C}$ of complex numbers is usually given as

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\},$$

where $i$ is a so-called “imaginary number”\(^1\) satisfying the property

$$i^2 = -1.$$

A more careful definition of the set of complex numbers is as follows:

$$\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\},$$

where $(a, b)$ here corresponds to what we usually think of as $a + bi$, so $(0, 1)$ corresponds to the term called $i$ above. Skipping over the question of multiplying complex numbers for the moment, let us see how addition of complex numbers and multiplication of complex numbers by real numbers work.

Addition of complex numbers is the same as the corresponding addition in $\mathbb{R}^2$:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

or $$(a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i.$$\(^1\)

1The terms “imaginary” and “complex”, contrasted with the term “real”, reflect the true historical reluctance with which complex numbers were received.
Also, multiplication of a complex number by a real number corresponds directly to scalar multiplication in $\mathbb{R}^2$: for $c \in \mathbb{R}$, 
$$c(a, b) = (ca, cb)$$
or 
$$c(a + bi) = ca + cbi.$$ 

Why do we bother defining complex numbers then, if they work just like $\mathbb{R}^2$? Well, they have something which vectors in $\mathbb{R}^2$ do not: complex numbers can be multiplied with each other. Multiplication of two complex numbers is defined by 
$$(a_1, b_1)(a_2, b_2) := (a_1a_2 - b_1b_2, a_1b_2 + b_1a_2)$$
or 
$$(a_1 + b_1i)(a_2 + b_2i) := (a_1a_2 - b_1b_2) + (a_1b_2 + b_1a_2)i.$$

The first definition of complex multiplication (the one without the $i$ in it) may seem somewhat mysterious at first, but when we look at the second definition, the motivation for this strange definition of multiplication should become more apparent. In particular, complex numbers multiply just as real polynomials in a single variable called $i$ would multiply, with the one exception that 
$$i^2 = -1.$$ 

This is the only relation that needs to be memorized to multiply complex numbers; all the other aspects of complex multiplication are already familiar from multiplying polynomials with real coefficients.

**Example 2.9.2** As an example of complex multiplication, we use the usual FOIL (“first, outside, inside, last”) method together with the fact that $i^2 = -1$ to multiply some complex numbers: 

$$(3 + 2i)(5 - i) = 15 - 3i + 10i - 2i^2$$
$$= 15 - 3i + 10i + 2$$
$$= 17 + 7i.$$ 

We use the same method in the following example:

$$(2 + i)(1 + 3i) = 2 + 6i + i + 3i^2$$
$$= 2 + 6i + i - 3$$
$$= -1 + 7i. \triangle$$

We have not yet defined division of complex numbers, but before doing so, we should first introduce some definitions and notation.

**Definition 2.9.3** For a complex number $z = a + bi$ with $a, b \in \mathbb{R}$, the number $a$ is called the real part of $z$, and the number $b$ is called the imaginary part of $z$. These are denoted by: 

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b.$$
Note that both the real part and the imaginary part of $z$ are defined to be real numbers.

**Definition 2.9.4** A complex number $z$ is called real if $\text{Im}(z) = 0$. A complex number is called imaginary (or pure imaginary by some authors) if $\text{Re}(z) = 0$.

**Definition 2.9.5** For any complex number $z = a + bi$, where $a, b \in \mathbb{R}$, the complex conjugate $\overline{z}$ of $z$ is defined to be

$$\overline{z} = a - bi.$$ 

The notation $z^*$ instead of $\overline{z}$ for the complex conjugate is also commonly used, especially in physics.

Notice that any number times its conjugate is real:

$$(a + bi)(a - bi) = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$ 

We can now give the definition of the magnitude of a complex number in terms of its complex conjugate.

**Definition 2.9.6** For any complex number $z = a + bi$, where $a, b \in \mathbb{R}$, the magnitude $|z|$ of $z$ (also called the length, norm, modulus or absolute value of $z$) is defined by

$$|z| = \sqrt{zz^*},$$ 

or, equivalently,

$$|z| = \sqrt{a^2 + b^2}.$$ 

This definition of length corresponds to the usual one used for vectors in $\mathbb{R}^2$, and consequently shares its various properties, such as being a nonnegative real number that is zero if and only if $z = 0$.

Since the notion of length carries over from $\mathbb{R}^2$ unchanged, so does the notion of distance. That is,

$$\text{Dist}(z_1, z_2) = |z_2 - z_1|.$$ 

This distance function has the usual distance properties, corresponding to those of the distance function in $\mathbb{R}^2$.

Elaborating on this notion, we find that since

$$z\overline{z} = a^2 + b^2$$

is zero if and only if $z = 0$, we can use this to define division of complex numbers.

**Definition 2.9.7** For any nonzero complex number $z \in \mathbb{C}$, we define

$$z^{-1} := \frac{1}{|z|^2}\overline{z}.$$
The fraction in this equation makes sense because \(|z|^2\) is a real number (and a nonzero one since \(z\) is assumed to be nonzero), so the division involved in it is simply the usual division in \(\mathbb{R}\).

If \(z = a + bi\), where \(a, b \in \mathbb{R}\), this definition becomes

\[
(a + bi)^{-1} := \frac{a}{a^2 + b^2} + \frac{b}{a^2 + b^2}i.
\]

Notice that by this definition,

\[
z \cdot z^{-1} = \frac{1}{|z|^2} z \overline{z} = \frac{1}{|z|^2} |z|^2 = 1,
\]

so \(z^{-1}\) really is the multiplicative inverse of \(z\), as the notation \(z^{-1}\) suggests.

From this, it is straightforward to define division of complex numbers.

**Definition 2.9.8** For any complex numbers \(z_1, z_2 \in \mathbb{C}\) with \(z_2 \neq 0\), the quotient of \(z_1\) by \(z_2\) is defined by:

\[
\frac{z_1}{z_2} := z_1 \overline{z_2}^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2}.
\]

Again the fact that \(|z|^2\) is a nonzero real number allows us to divide by it without any difficulty.

For any complex numbers \(z_1 = a_1 + b_1i\) and \(z_2 = a_2 + b_2i\), with \(a_1, b_1, a_2, b_2 \in \mathbb{R}\) with \(z_2 \neq 0\), this definition reads

\[
\frac{a_1 + b_1i}{a_2 + b_2i} := \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}i.
\]

**Example 2.9.9** We illustrate this definition by expressing the following quotients as \(a + bi\), where \(a, b \in \mathbb{R}\). Notice that, in effect, our method of doing so is to multiply both the numerator and denominator by the conjugate of the denominator.

\[
\frac{1}{3-i} = \frac{3 + i}{(3-i)(3+i)} = \frac{3 + i}{10} = \frac{3}{10} + \frac{1}{10}i.
\]

\[
\frac{2 - 3i}{1 + 2i} = \frac{(2-3i)(1-2i)}{(1+2i)(1-2i)} = \frac{8 - 7i}{5} = \frac{8}{5} - \frac{7}{5}i.
\]

\[
\frac{8 - i}{2 + 5i} = \frac{(8-i)(2-5i)}{(2+5i)(2-5i)} = \frac{11 - 42i}{29} = \frac{11}{29} - \frac{42}{29}i. \quad \blacksquare
\]

In short then, to “clear the imaginary part out of a denominator”, the method is to multiply both the numerator and the denominator by the conjugate of the
denominator. The result will be a real denominator, by which we can then divide in the usual way.

**Section 2.10: The geometry of complex numbers**

The very definition of complex numbers asserts that, as a set, they are the same as $\mathbb{R}^2$. Consequently, each complex number $a + bi$ can be viewed as corresponding to the vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, or simply to the point $(a, b)$ in the plane.

Since addition of complex numbers is defined just as it is correspondingly for vectors in $\mathbb{R}^2$, the associated picture of addition then is also the same.

![Figure 2.10.1: The sum $(a_1 + b_1i) + (a_2 + b_2i)$](image)

But the multiplication of complex numbers is new, and we should now explore it some. For this, it will be useful to shift to the point of view of polar coordinates.

Every vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ can be written in polar coordinates, namely as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix},$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan \left( \frac{b}{a} \right)$, as indicated in the picture below.
By the same reasoning (or perhaps merely a change in notation), any complex number \( a + bi \) can be written as

\[
a + bi = r(\cos \theta + i \sin \theta),
\]

where \( r \) and \( \theta \) are as defined above. This approach doesn’t seem to yield anything new at first glance, but it does if we take a little side trip first.

Recall that for any \( x \in \mathbb{R} \), the function \( e^x \) is defined by the power series

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},
\]

which converges for all values of \( x \in \mathbb{R} \). It turns out that the same power series

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}
\]

also converges for all values of \( z \in \mathbb{C} \), so we can extend the domain of the original function to include all complex numbers. This implies the following remarkable result, which we now prove.

**Theorem 2.10.2 (Euler’s Formula)**

\[
e^{i\theta} = \cos \theta + i \sin \theta \quad \text{for all } \theta \in \mathbb{R}.
\]
Proof Examining the defining power series, we find that
\[
e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \frac{i^{2n} \theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1} \theta^{2n+1}}{(2n+1)!}
\]
\[
= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}
\]
\[
= \cos \theta + i \sin \theta.
\]
In other words, when the power series for \(e^{i\theta}\) is split into its even and odd parts, the even part is precisely the power series for \(\cos \theta\), and the odd part is precisely \(i\) times the power series for \(\sin \theta\). (Those unconvinced by the fancy index shuffling above can write out the first handful of terms in the power series to see that this actually happens.)

The following corollary is immediate by substituting the value \(\pi\) in for \(\theta\) in Euler’s Formula, but it is often written separately since it expresses a wonderful relationship among five of the most significant numbers in all of mathematics: \(e, \pi, i, 1,\) and \(0\).

**Corollary 2.10.3**

\[e^{\pi i} + 1 = 0.\]

Returning to the question of the geometry of complex multiplication, we now know that the polar decomposition of a complex number can be written as
\[a + bi = r(\cos \theta + i \sin \theta) = re^{i\theta}.\]
It is this compact expression of the polar decomposition of a complex number which now allows us to view complex multiplication geometrically. We will need to assume one more thing which we have not proved, however, and that is that
\[e^{z_1}e^{z_2} = e^{z_1+z_2}\]
for all complex numbers \(z_1, z_2\). Proofs of this identity for real values of \(z\) carry over directly to the complex case though, so we will just take this identity as given.
In this case then, we have that

\[(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)},\]

for any complex numbers \(r_1 e^{i\theta_1}\) and \(r_2 e^{i\theta_2}\) expressed in polar form. We interpret this equation in the following proposition.

**Proposition 2.10.4**  The length of the product of two complex numbers is the product of their lengths. The polar angle of the product of two complex numbers is the sum of their polar angles.

**Proof**  See the equation immediately above the statement of the proposition, and note that

\[|re^{i\theta}| = |r \cos \theta + ir \sin \theta| = r.\]

Below is depicted an example of what complex multiplication looks like geometrically. The product of the numbers whose vectors are dashed equals the number whose vector is solid.

Note the way the lengths multiply and the angles add, in accordance with the above proposition.
1. Express the following in the form $a + bi$, where $a, b \in \mathbb{R}$.
   (a) $\frac{3 - i}{2 + 5i}$.
   (b) $\frac{4 + 3i}{1 + 2i}$.
   (c) $(\sqrt{3} + i)^{11}$. (Hint: think polar.)

2. Let $n$ be a positive integer. Find all complex numbers $z$, such that $z^n = 1$, and depict their location in a picture of the complex plane. (Hint: think polar, and try $n = 1, 2, 3, \ldots$ until you find the pattern.)

3. Find all complex numbers $z$, such that $z^3 = -8$.

4. Illustrate the following subsets of $\mathbb{C}$ in a picture of the complex plane.
   (a) $\{ z \in \mathbb{C} \mid |z - 3i| = 2 \}$.
   (b) $\{ z \in \mathbb{C} \mid z + \bar{z} = 0 \}$.
   (c) $\{ z \in \mathbb{C} \mid z + 5\bar{z} = 0 \}$.
   (d) $\{ z \in \mathbb{C} \mid |z + 2| = |z - 2i| \}$.
   (e) $\{ z \in \mathbb{C} \mid |z + (1 + i)| \leq 1 \}$.

5. (a) Show from the definition of length (namely, that $|z| = \sqrt{z\bar{z}}$) that $|z_1z_2| = |z_1||z_2|$ for all $z_1, z_2 \in \mathbb{C}$. (Do not resort to coordinates, polar or rectangular, here.)
   (b) Let $a_1, b_1, a_2, b_2 \in \mathbb{Z}$. Use Part (a) to find a formula expressing $(a_1^2 + b_1^2)(a_2^2 + b_2^2)$ as the sum of the squares of two integers. (This shows that the product of two numbers which are sums of perfect squares is again a sum of perfect squares.)

6. (a) Show that $\overline{(z_1z_2)} = \overline{z_1}\overline{z_2}$ for all $z_1, z_2 \in \mathbb{C}$.
   (b) Show that $|z| = |\overline{z}|$ for all $z \in \mathbb{C}$.

7. Use Euler’s Formula to give definitions of $\sin \theta$ and $\cos \theta$ in terms of $e^{\pm i\theta}$.
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8. (a) Check that for any nonzero integer \( n \),
\[
\int_0^{2\pi} e^{in\theta} \, d\theta = 0.
\]
(Hint: if you are not comfortable evaluating complex integrals, use Euler’s Formula to split this into real integrals.)

(b) Use Part (a) and the definitions from Problem 7 to compute
\[
\int_0^{2\pi} \cos^4 \theta \, d\theta.
\]
(This actually generalizes to a nice formula for \( \int_0^{2\pi} \cos^{2n} \theta \, d\theta \) if you chase it far enough.)