A graph is a collection of points (called vertices, plural of vertex), some of which may be connected by (straight or curved) paths (called edges). A graph is called connected if you can move along its edges from any vertex to any other vertex.

1. For connected graphs, there turns out to be a relationship between the number of vertices ($V$), the number of edges ($E$), and the number of faces ($F$), which are the regions completely enclosed by a set of edges. Find this relationship. (Hint: you will probably want to subtract one of these from the other two.) Draw at least three graphs that helped you find this relationship, and give $V, E, F$ for all three of them.

2. Two vertices are defined to be in the same component of a graph if you can move from one to the other along the edges in the graph. Let $C$ be the number of components of a graph. Find a relationship between $V, E, F, C$. Draw at least three graphs that helped you find this relationship, and give $V, E, F, C$ for all three of them.

3. An Euler path on a graph is a path (starting at a vertex and ending at a vertex) that traverses each edge exactly once. Determine a simple way to tell if a connected graph has an Euler path, and explain why it works. (You don’t need to provide a rigorous mathematical proof, but give some justification for the condition that you give.) As a hint, you might think about the number of edges emanating from each vertex.

4. An Euler cycle on a graph is a path that starts and ends at the same vertex and that traverses each edge exactly once. Determine a simple way to tell if a connected graph has an Euler cycle, and explain why it works. (You don’t need to provide a rigorous mathematical proof, but give some justification for the condition that you give.)