Klein’s *Erlangen program* of 1872 was a major landmark in the history of geometry:

- It made sense of the zoo of geometries that had appeared
- It organized the known geometries
- It answered the question, *What is a geometry?*
- It closed the door on haphazard and ad hoc development of new geometries

It brought this line of thinking about Euclidean geometries to a close.
Some of the geometries that had appeared:

- Euclidean
- spherical
- hyperbolic
- projective

These are summarized at http://www.cmis.brighton.ac.uk/staff/jt40/EM225/EM225_ComparisonsOfGeometries.pdf
In his Erlangen program, Klein defined a geometry as a set $S$ together with a group $G$ of transformations from $S$ to $S$.

The geometric properties within a geometry are those properties that are unchanged by all the transformations in the group.

Classic example: Euclidean plane geometry consists of the set of points in the plane together with the group of rigid transformations (or isometries).

Length, distance, and angle are all geometric properties of Euclidean geometry (they are left unchanged by all rigid transformations).

Other geometries can be viewed similarly, including inversive and affine geometries.
If two geometries have the same point set, but the transformation group $H$ for one is a subgroup of the other’s $G$, everything that is a property of the $G$ geometry is automatically a property for the $H$ geometry.

That is, a smaller transformation group means more geometric properties.
Building on ideas developed by Gauss and Riemann, Klein applied his group theoretic definition of a geometry more generally to “locally Euclidean” spaces.

In modern terms, Klein considered manifolds with a group acting on them.

Riemann, however, preferred to consider manifolds with a notion of distance.
Klein’s definition of a geometry is broader: invariants of the transformation group don’t need to include distance

However, by requiring a little bit more, Riemann’s approach has turned out to be more effective in a variety of contemporary mathematical settings

The two use different approaches to determining equivalence of geometric objects
Equivalence relations are quite common in contemporary mathematics.

An equivalence relation on a set \( S \) is a relation \( \sim \) that can be used to compare any two elements of \( S \) that is:

1. **reflexive**, meaning \( a \sim a \) (for all \( a \) in \( S \)),
2. **symmetric**, meaning that if \( a \sim b \) then \( b \sim a \) (for all \( a, b \) in \( S \))
3. **transitive**, meaning that if \( a \sim b \) and \( b \sim c \), then \( a \sim c \) (for all \( a, b, c \) in \( S \))
Example of equivalence relations: equals, congruence, similar triangles, etc.

Example of relations that are not equivalence relations: greater than, less than
An equivalence relation on a set $S$ partitions $S$ into *equivalence classes*. 

Two elements $a, b$ in $S$ are defined to be in the same equivalence class if $a \sim b$.

When we speak of a geometric object, we are really speaking of the equivalence class of a set of points, where two sets of points are equivalent if one can be taken to the other by a rigid transformation.
The mathematical field of *topology* is related to geometry (in both Klein’s and Riemann’s sense), but it is slightly different.

It can be thought of as the study of *shape*.

Roughly speaking, two topological spaces are equivalent if one can be continuously deformed into the other.

(The actual mathematical requirement is that there is a continuous function from one to the other that has a continuous inverse.)
Graph theory could be considered 1-dimensional topology (which began with Euler’s solution to the Königsberg bridge problem).

Another type of 1-dimensional topology is knot theory.

A knot is a closed loop in 3-dimensional space.

A link is a collection of non-intersecting closed loops in 3-dimensional space.

Two knots (or links) are equivalent if one can be deformed into the other without any self-crossings in the process.

“Deformed” does include stretching and shrinking, but does not include crossing through itself.
A big mathematical theorem is that two knots (or links) are equivalent if and only if one diagram can be made to coincide with the other through a series of the three *Reidemeister moves*

For these, see the Wikipedia article: https://en.wikipedia.org/wiki/Knot_theory

The central problem in knot theory: how can we tell (perhaps from looking at the diagrams) whether or not two knots (or links) are equivalent?
If you can find a sequence of Reidemeister moves (or a 3-dimensional manipulation) that takes one knot diagram to another, then the knots are equivalent.

What if you can’t? Then either they are not equivalent, or you just haven’t searched hard enough.

To demonstrate that two knots (or links) are not equivalent, we can use invariants.

An invariant is a quantity related to a knot (or link) that doesn’t change under the allowable deformations.
A simple example of an invariant is the number of components in a link.

A more complicated example: the linking number.

To compute the linking number of two components in a link, orient both components, add up the positive and negative crossings, and divide by two.

Positive crossings are where the lower strand crosses under the upper strand from right to left.

Negative crossings are where the lower strand crosses under the upper strand from left to right.
But the linking number is an invariant for pairs of link components

An invariant of knots is *tricolorability*:
https://en.wikipedia.org/wiki/Tricolorability
The *unknotting number* of a knot is easy to define: it is the minimum number of times a knot must cross through itself to become unknotted.

While this is easy to define, it is in general very difficult to compute!
We have seen some 1-dimensional topology, but what about 2-dimensional topology?

A couple of interesting 2-dimensional topological spaces: a Möbius strip and a Klein bottle
Both of these are 1-sided

What happens if we cut a Möbius strip down the middle? More than once? Interesting things.

What if we cut a Klein bottle open?
Friday, Mar. 31

It turns out that closed orientable surfaces (surfaces without a boundary) can be classified by counting “holes”

The number of “holes” is called the genus

However, sometimes a shape is sufficiently complicated that it is difficult to count the holes (such as when you drill through the center square in all sides of a Rubik’s cube)

Strangely enough, graph theory comes to the rescue here: we triangulate the space
A triangulation is a covering of the surface by (topological) triangles in such a way that the intersection of any pair of triangles is exactly one of the following three sets:

1. the empty set (non-intersecting)
2. a single point that is a corner (vertex) on both triangles
3. a (topological) line segment that is a side (edge) on both triangles

Halfway-overlapping triangles are not allowed in a triangulation, for example
Now start writing down $V$ (number of vertices), $E$ (number of edges), and $F$ (number of faces) on your favorite surfaces.

What is the relationship between the number of holes ($g$) and the other three quantities above?

The quantity $V - E + F$ is called the Euler characteristic of the surface, denoted by $\chi$ (the Greek letter chi).

It does not depend on which triangulation is used, and it is a topological invariant.

It is related to the genus, and you will investigate this on the upcoming homework assignment.