Homework 9

E1. In $\mathbb{R}^2$, find a formula for (and illustrate with a picture) the reflection of a vector $v \in \mathbb{R}^2$ across (the line spanned by) a given nonzero vector $w \in \mathbb{R}^2$. Your formula should be in terms of $\text{Proj}_w(v)$.

Since this formula in Part (a) is expressed in terms of orthogonal projection, we will take it as the definition of reflection across a nonzero vector in any real inner product space.

E2. For a given nonzero vector $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection across $w$, as defined in the previous problem.

(a) Prove that $T$ is a linear transformation.

(b) Find $T_{e_i,e_i}$.

(c) Since $w \neq 0$, then $\{w\}$ is linearly independent. This means that $\{w\}$ can be extended to an orthogonal basis for $\mathbb{R}^3$, say to $B = (w, b_2, b_3)$.

Find $T_{B,B}$.

E3. Let $n$ be a positive integer, and let $x_1, \ldots, x_n \in \mathbb{R}$. Prove that

$$\left( \frac{1}{n} \sum_{i=1}^{n} x_i \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} x_i^2.$$  

Another way to state this as that the square of their mean is less than or equal to the mean of their squares.  

*Hint:* Apply the Cauchy-Schwarz inequality to the dot product of two particular vectors in $\mathbb{R}^n$.

E4. Apply the Gram-Schmidt orthonormalization process to $A = ( \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} ) \subseteq \mathbb{R}^3$
to obtain an orthonormal basis for $\mathbb{R}^3$ whose partial spans are the same as the partial spans of $A$.

E5. Let

$$B = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \}. $$

Find $\begin{bmatrix} 1 \\ 49 \\ 9 \\ 29 \end{bmatrix}_B$.