Homework 12

E1. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be reflection across the vector \( \mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \). Without writing down a matrix representation of \( T \), explain geometrically what the eigenvalues of \( T \) are and what the associated eigenspaces of those eigenvalues are.

E2. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be given by
\[
T = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}^{\mathcal{E},\mathcal{E}}.
\]
By the spectral theorem for self-adjoint operators, \( T \) has an eigenbasis. Find an ordered eigenbasis \( B \), stating the eigenvalue for each eigenvector in \( B \). Also, find \( I_{\mathcal{E},B} \) and \( I_{B,\mathcal{E}} \) (and check for yourself that \( T_{B,B} = I_{B,\mathcal{E}} T_{\mathcal{E},\mathcal{E}} I_{\mathcal{E},B} \)).

E3. Let \( T : \mathbb{R}^4 \to \mathbb{R}^4 \) be given by
\[
T = \begin{bmatrix}
35 & -13 & -7 & -3 \\
-13 & 35 & -7 & -3 \\
-7 & -7 & 29 & -3 \\
-3 & -3 & -3 & 21
\end{bmatrix}^{\mathcal{E},\mathcal{E}}.
\]
(a) Find all the eigenvalues of \( T \) and their associated eigenspaces. Does \( T \) have an eigenbasis? If so, find \( T_{B,B} \) for an ordered eigenbasis \( B \). If not, explain why not.

(b) Repeat Part (a) for \( T \) viewed as a linear transformation \( T : \mathbb{C} \to \mathbb{C} \).

Let \( T : \mathbb{R}^4 \to \mathbb{R}^4 \) be given by
\[
T = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}^{\mathcal{E},\mathcal{E}}.
\]
E4. (a) Find all the eigenvalues of $T$ and their associated eigenspaces. Does $T$ have an eigenbasis? If so, find $T_{B,B}$ for an ordered eigenbasis $B$. If not, explain why not.

(b) Repeat Part (a) for $T$ viewed as a linear transformation $T : \mathbb{C} \to \mathbb{C}$.

E5. Let $P_2$ be the real vector space of polynomials in $x$ of degree at most 2. Let $T : P_2 \to P_2$ be differentiation with respect to $x$, so

$$T(p(x)) = p'(x).$$

We have seen that $T$ is a linear transformation. Find all its eigenvectors and their associated eigenspaces. Does $T$ have an eigenbasis?

E6. Let $V$ be the real vector space of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. You don’t need to prove it here, but differentiation $T : V \to V$ is a linear operator on $V$:

$$T(f(x)) = f'(x)$$

for all infinitely differentiable $f : \mathbb{R} \to \mathbb{R}$. Using what you know from calculus about differentiation, find all the eigenvalues of $T$ and give their associated eigenspaces.