Problems

1. Let

\[
T = \begin{bmatrix}
1 & -4 & -1 & 1 \\
2 & -8 & 1 & 4 \\
-1 & 4 & -2 & 5
\end{bmatrix}, \quad x = \begin{bmatrix}
 x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}_{C,B}, \quad b = \begin{bmatrix}
 3 \\
 9
\end{bmatrix}_{C}
\]

Also, suppose that for a reduced row echelon basis \(D\):

\[
T_{D,B} = \begin{bmatrix}
1 & -4 & 0 & -1 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad b_D = \begin{bmatrix}
 4 \\
 1 \\
 0
\end{bmatrix}
\]

(a) What is a basis for \(\text{Image}(T)\), represented with respect to \(D\)?

(b) What is a basis for \(\text{Image}(T)\), represented with respect to \(C\)?

(c) What is the solution set for \(x\) to the equation \(T(x) = b\), represented with respect to \(B\)?

(d) What is a basis for \(\text{Ker}(T)\), represented with respect to \(B\)?

(e) Suppose that, instead of the given \(b_D\), we have:

\[
b_D = \begin{bmatrix}
 5 \\
 -2 \\
 4
\end{bmatrix}.
\]

What is the solution set for \(x\) to the equation \(T(x) = b\), represented with respect to \(B\)?

2. Let

\[
T_{C,B} = \begin{bmatrix}
2 & -2 & -1 & 6 & -2 \\
1 & -1 & 1 & 2 & -1 \\
4 & -4 & 5 & 7 & -1
\end{bmatrix}, \quad b_C = \begin{bmatrix}
 1 \\
 2 \\
 6
\end{bmatrix}
\]

Also, suppose that for a reduced row echelon basis \(D\):

\[
T_{D,B} = \begin{bmatrix}
1 & -1 & 0 & 0 & 23 \\
0 & 0 & 1 & 0 & -6 \\
0 & 0 & 0 & 1 & -9
\end{bmatrix}, \quad b_D = \begin{bmatrix}
 -23 \\
 7 \\
 9
\end{bmatrix}
\]
3. Let
\[
T = \begin{bmatrix}
1 & 2 & -1 \\
2 & 2 & 1 \\
3 & 5 & -2
\end{bmatrix}
\quad C_B,
\quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\quad B
\quad b = \begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix}
\quad C
\]

Also, let \( D \) be a reduced row echelon basis relative to \( T \) and \( B \).

(a) What are \( T_D, B \) and \( b_D \)?
(b) What is a basis for \( \text{Image}(T) \), represented with respect to \( D \)?
(c) What is a basis for \( \text{Image}(T) \), represented with respect to \( C \)?
(d) What is the solution set for \( x \) to the equation \( T(x) = b \), represented with respect to \( B \)?
(e) What is a basis for \( \text{Ker}(T) \), represented with respect to \( B \)?

4. Let
\[
T = \begin{bmatrix}
3 & -1 & 2 & 4 & 1 \\
1 & -1 & 2 & 3 & 1 \\
2 & -3 & 6 & 9 & 4 \\
7 & -2 & 4 & 8 & 1
\end{bmatrix}
\quad C_B,
\quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
\quad B
\quad b = \begin{bmatrix}
2 \\
-1 \\
-5 \\
6
\end{bmatrix}
\quad C
\]

Also, let \( D \) be a reduced row echelon basis relative to \( T \) and \( B \).

(a) What are \( T_D, B \) and \( b_D \)?
(b) What is a basis for \( \text{Image}(T) \), represented with respect to \( D \)?
(c) What is a basis for \( \text{Image}(T) \), represented with respect to \( C \)?
(d) What is the solution set for \( x \) to the equation \( T(x) = b \), represented with respect to \( B \)?
(e) What is a basis for \( \text{Ker}(T) \), represented with respect to \( B \)?